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CONSTRUCTIVE APPROXIMATION
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Bases in the Spaces of C^∞-Functions on Cantor-Type Sets

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Abstract. We construct a topological basis in the space of Whitney functions given on the Cantor-type set.

1. Introduction

The basis problem is one of the most important parts of the theory of the structure of functional spaces. Are the spaces isomorphic? Do they have certain linear topological properties? Investigating these and some other questions is simpler when we consider the spaces of basis expansions of elements. On the other hand, the functions that form a basis in the functional space X, as usual, play a special role in the concrete problems of analysis related to X. One can mention here the Chebyshev polynomials in the space $C^\infty[-1, 1]$ [15], the Hermite functions in the space S of rapidly decreasing C^∞-functions [15], the Faber polynomials in spaces of analytic functions (see, e.g., [16]), and the Franklin sequence in the Hardy space H^1 [17]. Wavelets, widely used for diverse scientific applications, form unconditional bases in a variety of functional spaces on $\mathbf{R}^n$ (see, e.g., [11] and [18]).

In the case where $X(K)$ is the space of traces on a compact set $K \subset \mathbf{R}^n$ of functions from the certain class $X(\mathbf{R}^n)$, we can construct a continuous linear extension operator $L : X(K) \to X(\mathbf{R}^n)$ by means of suitable extensions of the basis elements of $X(K)$. This method goes back to Mityagin [15] and was, for example, used in the case of the spaces of Whitney functions in [9] and for ultradifferentiable functions in [2].

Among many results on the existence or lack of a basis in the spaces of holomorphic or differentiable functions, there are two related to the basis problem in the case of C^∞-functions given on Cantor-type sets. In [19] Zeriahi proved the existence of a measure, such that the polynomials orthogonal with respect to this measure form a basis in the Whitney space $\mathcal{E}(K)$ when the compact set K has the Markov property. In particular, the classical Cantor ternary set satisfies the condition as proved in [3]. Moreover, by Proposition 2 in [4] the Cantor-type set has Markov's property if and only if it is uniformly perfect.

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Using another technique, Keşir and Kocatepe proved in [14] the existence of a basis in the space $\mathcal{E}(K)$ for the Cantor-type set K with the extension property, that is, where there exists a continuous linear extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbf{R})$. Geometrically this means that the Cantor-type set is not very rarefied.

Here we present explicitly a Schauder basis in the space $\mathcal{E}(K)$ for any Cantor-type set. In the case of rarefied sets K, the partial sums of the basic expansion of $f \in \mathcal{E}(K)$ are just the interpolating polynomials of f corresponding to the uniform distribution of nodes on K. In another case, which includes the classical Cantor set, local interpolations of functions will be used.

It should be noted that interpolation Schauder bases in other functional spaces on fractals were given in [12] and [13]. For wavelets on fractals, see, for instance, [6].

2. Biorthogonal Systems

Given a compact set $K \subset \mathbf{R}$ and a sequence of distinct points $(x_n)_1^\infty \subset K$, let $e_n(x) = \prod_1^n (x - x_k)$ for $n \in \mathbf{N}_0 := \{0, 1, \ldots\}$. Here and in what follows we adopt the convention that $\prod_m^n(\cdots) = 1$ for $m > n$. Let $X(K)$ be a Fréchet space of continuous functions on K, containing all polynomials. By ξ_n we denote the linear functional $\xi_n(f) = [x_1, x_2, \ldots, x_{n+1}]f$ with $f \in X(K)$ and $n \in \mathbf{N}_0$. For the definition and properties of the divided differences, see, e.g., [5]. We have, trivially,

Lemma 1. *If a sequence $(x_n)_1^\infty$ of distinct points is dense on a perfect compact set $K \subset \mathbf{R}$, then the system $(e_n, \xi_n)_{n=0}^\infty$ is biorthogonal and the sequence of functionals $(\xi_n)_{n=0}^\infty$ is total on $X(K)$, that is, whenever $\xi_n(f) = 0$ for all n, it follows that $f = 0$.*

As in [8] we will consider different basic systems and the following convolution property of the coefficients of basis expansions.

Lemma 2. *Let $(x_k^{(s)})_{k=1}^\infty$, $s = 1, 2, 3$, be three sequences such that for a fixed superscript s all points in the sequence $(x_k^{(s)})_{k=0}^\infty$ are different. Let $e_{ns} = \prod_{k=1}^n (x - x_k^{(s)})$ and $\xi_{ns}(f) = [x_1^{(s)}, x_2^{(s)}, \ldots, x_{n+1}^{(s)}]f$ for $n \in \mathbf{N}_0$. Then*

$$\sum_{q=p}^{r} \xi_{p3}(e_{q2})\, \xi_{q2}(e_{r1}) = \xi_{p3}(e_{r1}) \qquad \text{for} \quad p \le r.$$

Proof. We have three bases $(e_{ns})_{n=0}^r$, $s = 1, 2, 3$, in the $(r+1)$-dimensional vector space $\prod_r$ of all polynomials of a degree less than or equal to r. If $M_{i \leftarrow j}$ denotes the transition matrix from the jth basis of $\prod_r$ to the ith one, then $\xi_{p3}(e_{r1})$ gives the (p, r)th element of $M_{3 \leftarrow 1}$ which equals $M_{3 \leftarrow 2} M_{2 \leftarrow 1}$. ■

By means of Lemma 2 we can construct new biorthogonal systems corresponding to the local interpolation of functions. Suppose we have a chain of compact sets $K_0 \supset K_1 \supset \cdots \supset K_s \supset \cdots$ and finite systems of distinct points $(x_k^{(s)})_{k=1}^{N_s} \subset K_s$ for $s = 0, 1, \ldots$. Some part of the knots on K_{s+1}—let $(x_k^{(s+1)})_{k=1}^{M_{s+1}}$—belongs to the previous set

$(x_k^{(s)})_{k=1}^{N_s}$. The sequences (M_s) and (N_s) can be specified later. In what follows we will take $2M_{s+1} = N_s \le N_{s+1}$.

For any $s \ge 0$ and for $n = M_s + 1, \dots, N_s$, set $e_{ns} = \prod_{k=1}^{n}(x - x_k^{(s)})$ for $x \in K_s$ and $e_{ns} = 0$ for $x \in K_0 \backslash K_s$. If $K_{s-1} \backslash K_s$ is closed for any $s \ge 1$, then the functions e_{ns} are continuous on K_0. Let $\xi_{ns}(f) = [x_1^{(s)}, x_2^{(s)}, \dots, x_{n+1}^{(s)}]f$ with $x_{N_s+1}^{(s)} := x_{M_{s+1}+1}^{(s+1)}$. We see at once that $\xi_{ns}(e_{m,s+1}) = 0$, because the number $\xi_{ns}(f)$ is defined by values of f at some points on $K_s \backslash K_{s+1}$ and at some points from $(x_k^{(s+1)})_{k=1}^{M_{s+1}}$, where the function $e_{m,s+1}$ is zero. Clearly, $\xi_{n,s+1}(e_{ms}) = 0$ for $n > m$. But, for $n \le m$, the functional $\xi_{n,s+1}$, in general, is not biorthogonal to e_{ms}. For this reason we take the functional

$$\eta_{n,s+1} = \xi_{n,s+1} - \sum_{k=n}^{N_s} \xi_{n,s+1}(e_{ks})\xi_{ks},$$

which is biorthogonal, not only to all elements e_{ms}, but also, by the convolution property, to all e_{mj} with $j = 0, 1, \dots, s-1$.

3. Rarefied Cantor Sets

Let $\Lambda = (l_s)_{s=0}^{\infty}$ be a sequence such that $l_0 = 1$ and $0 < 3l_{s+1} \le l_s$ for $s \in \mathbf{N}_0$. Let $K(\Lambda)$ be the Cantor set associated with the sequence Λ, that is, $K(\Lambda) = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0, 1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,s}$ of length l_s and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,s}$, $j = 1, 2, \dots, 2^s$. Denote $\alpha_s = \log l_{s+1}/\log l_s$ for $s \in \mathbf{N}$. Thus, $l_{s+1} = l_1^{\alpha_1 \dots \alpha_s}$. Let x be an endpoint of some basic interval. Then there exists the minimal number s (the *type* of x) such that x is the endpoint of some $I_{j,m}$ for every $m \ge s$.

Let us choose the sequence $(x_n)_1^{\infty}$ by including all endpoints of basic intervals, using the rule of increase of the type. For points of the same type we first take the endpoints of the largest gaps between the points of this type; here the intervals $(-\infty, x)$, (x, ∞) are considered as gaps. From points adjacent to the equal gaps, we choose the left one x and then $1 - x$. Thus, $x_1 = 0, x_2 = 1, x_3 = l_1, \dots, x_7 = l_1 - l_2, \dots, x_{2^k+1} = l_k, \dots$.

We consider the space $\mathcal{E}(K(\Lambda))$ of Whitney functions on $K(\Lambda)$ with the topology defined by the norms

$$\|f\|_q = |f|_q + \sup\{|(R_y^q f)^{(i)}(x)| \cdot |x - y|^{i-q}; x, y \in K(\Lambda), x \ne y, i = 0, 1, ..., q\},$$

$q = 0, 1, \dots,$ where $|f|_q = \sup\{|f^{(i)}(x)| : x \in K(\Lambda), i \le q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder. Each function $f \in \mathcal{E}(K(\Lambda))$ is extendable to a C^∞-function on the line. Since the compact set $K(\Lambda)$ is perfect, the set $(f^{(i)}(x))_{i \in \mathbf{N}_0, x \in K(\Lambda)}$ is completely defined by the values of f on $K(\Lambda)$.

Let $e_N(x) = \prod_1^N (x - x_k)$ and $\xi_N(f) = [x_1, x_2, \dots, x_{N+1}]f$ for $N \in \mathbf{N}_0$.

Theorem 1. *For a sequence Λ let us have $\alpha_s \ge 2$, $s \in \mathbf{N}$. Then the sequence $(e_N)_{N=0}^{\infty}$ is a Schauder basis in the space $\mathcal{E}(K(\Lambda))$.*

Proof. By Lemma 1 the system $(e_N, \xi_N)_{N=0}^{\infty}$ is biorthogonal with a total sequence of functionals. Therefore, by the Dynin–Mityagin criterion [15, T.9], it is enough to show that for every p there exist r and C such that, for all N,

$$\|e_N\|_p \cdot |\xi_N|_{-r} \le C.$$

Here, and subsequently, $|\cdot|_{-r}$ denotes the dual norm: for $\xi \in \mathcal{E}'(K)$ let $|\xi|_{-r} = \sup\{|\xi(f)|,\ \|f\|_r \le 1\}$.

There is no loss of generality in assuming that $p = 2^u$. Given u, we take $q = 2^v - 1$, where $v = v(u)$ and $r = r(q)$ will be specified later. Let us fix $N = 2^n + \nu$, where $0 \le \nu = 2^{r_1} + \cdots + 2^{r_m} < 2^n$ with $0 \le r_m < \cdots < r_1 < r_0 := n$. According to the procedure we choose at first all 2^n points of the type less than or equal to $n-1$. The remaining ν points of nth type we separate into groups: 2^{r_j} points (let us denote this set by X_{r_j}) are uniformly distributed on the basic intervals I_{m,r_j}, $m = 1, 2, \ldots, 2^{r_j}$. If $\nu = 0$, then sets X_{r_j} are empty for $j \ge 1$. In this notation $e_N(x) = \prod_{j=0}^m \prod_{x_k \in X_{r_j}} (x - x_k)$.

Every interval of length l_{r_j} contains just one point from the set X_{r_j}. By the structure of the set $K(\Lambda)$, for $x \in K(\Lambda)$ we get $\prod_{x_k \in X_{r_j}} |x - x_k| \le l_{r_j} l_{r_j-1} l_{r_j-2}^2 \cdots l_0^{2^{r_j-1}}$. Therefore,

$$|e_N|_0 \le \prod_{j=0}^{m} (l_{r_j} l_{r_j-1} l_{r_j-2}^2 \cdots l_0^{2^{r_j-1}}) = \prod_{1}^{N} z_k, \tag{1}$$

where $(z_k)_1^N$ are arranged in nondecreasing order. For example, if $\nu < 2^{n-1}$, then $z_1 = l_n, z_2 = l_{n-1}, z_3 = z_4 = l_{n-2}, \ldots$; if $\nu \ge 2^{n-1}$, then $z_1 = l_n, z_2 = z_3 = l_{n-1}, z_4 = l_{n-2}, \ldots$.

Arguing as in [10, L.2] we get, for $N > p$,

$$|e_N|_p \le N^p \prod_{p+1}^{N} z_k.$$

In order to estimate $\|e_N\|_p$, let us fix $x, y \in K$, $i \le p$. Let R denote $(R_y^p e_N)^{(i)}(x)$. Suppose, at first, that x and y belong to the same basic interval $I_{j,n-u+1}$. By the Lagrange form of the Taylor remainder, $|R| \cdot |x-y|^{i-p} \le |e_N^{(p)}(\theta) - e_N^{(p)}(y)|$, where $\theta \in I_{j,n-u+1}$. As above we get the bound $|e_N^{(p)}(\theta)| \le N^p \prod_{p+1}^N d_k(\theta)$ with $d_k(\theta) := |\theta - x_{i_k}| \uparrow$. The interval $I_{j,n-u+1}$ contains λ points (with $p/2 \le \lambda \le p$) of the set $(x_k)_1^N$. But $d_k(\theta) \le z_k$ for $k > \lambda$. Therefore, $|R| \cdot |x-y|^{i-p} \le 2N^p \prod_{p+1}^N z_k$.

Suppose now that $|x-y| \ge h_{n-u} \ge \frac{1}{3} l_{n-u}$. Then, for any j with $i \le j \le p$, we get the bound

$$|e_N^{(j)}(y)| \cdot |x-y|^{j-p} \le 3^p l_{n-u}^{j-p} N^j \prod_{j+1}^{N} z_k \le 3^p N^p \prod_{p+1}^{N} z_k,$$

as every interval of the length l_{n-u} contains not less than p points from $(x_k)_1^N$ and $z_{j+1}, \ldots, z_p \le l_{n-u}$.

Hence, $|R| \cdot |x-y|^{i-p} \le |e_N^{(i)}(x)| \cdot |x-y|^{i-p} + \sum_{j=i}^p |e_N^{(j)}(y)| \cdot |x-y|^{j-p}/(j-i)! \le (e+1)(3N)^p \prod_{p+1}^N z_k$. Thus,

$$||e_N||_p \le 5(3N)^p \prod_{p+1}^{N} z_k.$$

To estimate the dual qth norm of ξ_N we suppose that N is large enough, enumerate the first $N+1$ points of the sequence $(x_n)_1^\infty$ in increasing order, and use the bound (1) from [10]:

$$(2) \qquad |[x_1, \ldots, x_{N+1}]f| \le 2^{N-q} |\tilde{f}|_q^{([0,1])} \left(\min \prod_{m=1}^{N-q} |x_{a(m)} - x_{b(m)}| \right)^{-1},$$

where $\tilde{f} \in C^\infty[0,1]$ is any extension of f on $[0,1]$; min is taken over all $1 \le j \le N+1-q$ and all possible chains of strict embeddings $[x_{a(0)}, \ldots, x_{b(0)}] \subset [x_{a(1)}, \ldots, x_{b(1)}] \subset \cdots \subset [x_{a(N-q)}, \ldots, x_{b(N-q)}]$ with $a(0) = j,\ b(0) = j+q, \ldots, a(N-q) = 1,\ b(N-q) = N+1$. Here, given $a(k),\ b(k)$ we take $a(k+1) = a(k),\ b(k+1) = b(k)+1$ or $a(k+1) = a(k)-1,\ b(k+1) = b(k)$. We will denote by $\prod$ the minimizing product above.

Let us consider all possible locations of $q+1$ consecutive points $(x_{j+k})_{k=0}^q$ from $(x_n)_1^{N+1}$. Every interval of the length l_{n-v} contains more than 2^v such points. Therefore the product above can take its minimal value only if all $q+1$ points are situated on the same interval of this length. Fix this interval $I_{i,\,n-v}$. Let it contain μ points from $(x_n)_1^{N+1}$. Each of the two subintervals $I_{2i-1,\,n-v+1}$, $I_{2i,\,n-v+1}$ of $I_{i,\,n-v}$ contains at most 2^v points, therefore the first $\mu - q - 1$ terms of the product $\prod$ are larger than the length of the gap h_{n-v}. Other terms of $\prod$ can be estimated from below by the lengths of the gaps $h_{n-v-1}, h_{n-v-2}, \ldots, h_0$. Hence we get the product as in (1), but l_k should be replaced by h_k and the smallest q terms are absent. Since $h_k/l_k = 1 - 2\,l_{k+1}/l_k \ge 1 - 2\,l_1$, as $l_k \searrow$ and $\alpha_k \ge 2$, therefore, $\prod \ge (1-2\,l_1)^{N-q} \cdot \prod_{q+1}^N z_k \ge l_1^N \cdot \prod_{q+1}^N z_k$.

In addition (see [10, T.1] for more details), by the open mapping theorem for a given q, there exists $r \in \mathbf{N},\ C_q > 0$ such that

$$(3) \qquad \inf |\tilde{f}|_q^{([0,1])} \le C_q ||f||_r$$

for any $f \in \mathcal{E}(K(\Lambda))$. Here inf is taken over all possible extensions of f to $\tilde{f}$ on [0, 1]. This yields

$$\|e_N\|_p \cdot |\xi_N|_{-r} \le C 2^N l_1^{-N} N^p \prod_{p+1}^q z_k,$$

where $C = 5C_q 3^p$.

For the estimation of the product $\prod_{p+1}^q z_k$ let us take into account only the terms z_k corresponding to the points from the set X_{r_0}. Clearly, including the points from other sets X_{r_j} can only decrease this product. Thus we have to remove p smallest terms of the product $l_n l_{n-1} l_{n-2}^2 \cdots l_{n-v+1}^{2^{v-2}} l_{n-v}^{2^{v-1}-1}$. Neglecting the last term we get $\prod_{p+1}^q z_k \le l_{n-u-1}^{2^u} l_{n-u-2}^{2^{u+1}} \cdots l_{n-v+1}^{2^{v-2}} = l_1^\kappa$ with

$$\kappa = 2^u \alpha_1 \cdots \alpha_{n-u-2} + \cdots + 2^{v-2} \alpha_1 \cdots \alpha_{n-v} \ge 2^{n-2}(v-u-1),$$

as $\alpha_s \ge 2$. Taking into account the bound $N < 2^{n+1}$, we obtain

$$\|e_N\|_p \cdot |\xi_N|_{-r} \le C(2/l_1)^{2^{n+1}} 2^{(n+1)\,p} l_1^{2^{n-2}(v-u-1)}.$$

The value v such that $(v-u-1)\ln 1/l_1 > 8 \ln 2/l_1$ gives the desired conclusion, as is easily checked. ■

4. Local Interpolation

In the case of the space $\mathcal{E}(K(\Lambda))$, with $\alpha_s < 2$, $s \in \mathbf{N}$, the condition of the Dynin–Mityagin criterion is not valid for the system $(e_N, \xi_N)_{N=0}^\infty$, so we have to modify it.

Given a nondecreasing sequence of natural numbers $(n_s)_0^\infty$, let $N_s = 2^{n_s}$, $M_s^{(l)} = N_{s-1}/2 + 1$, $M_s^{(r)} = N_{s-1}/2$ for $s \geq 1$ and $M_0 = 0$. Here, (l) and (r) mean *left* and *right*, respectively. For the fixed basic interval $I_{j,s} = [a_{j,s}, b_{j,s}]$ we choose the sequence of points $(x_{n,j,s})_{n=1}^\infty$ using the procedure described in Section 3. Of course, instead of $1 - x$, we will take $b_{j,s} + a_{j,s} - x$.

Set $e_{N,1,0} = \prod_{n=1}^N (x - x_{n,1,0}) = \prod_1^N (x - x_n)$ for $x \in K(\Lambda)$, $N = 0, 1, \ldots, N_0$. For $s \geq 1$, $j \leq 2^s$ let $e_{N,j,s} = \prod_{n=1}^N (x - x_{n,j,s})$ if $x \in K(\Lambda) \cap I_{j,s}$ and $e_{N,j,s} = 0$ on $K(\Lambda)$ otherwise. Here, $N = M_s^{(a)}, M_s^{(a)} + 1, \ldots, N_s$ with $a = l$ for odd j and $a = r$ if j is even. Biorthogonal functionals are given in the following way: for $s = 0, 1, \ldots, j = 1, 2, \ldots, 2^s$, and $N = 0, 1, \ldots$, let $\xi_{N,j,s}(f) = [x_{1,j,s}, \ldots, x_{N+1,j,s}]f$. Set $\eta_{N,1,0} = \xi_{N,1,0}$ for $N \leq N_0$. Every basic interval $I_{j,s}$, $s \geq 1$, is a subinterval of a certain $I_{i,s-1}$ with $j = 2i - 1$ or $j = 2i$. Let

$$\eta_{N,j,s}(f) = \xi_{N,j,s}(f) - \sum_{k=N}^{N_{s-1}} \xi_{N,j,s}(e_{k,i,s-1})\,\xi_{k,i,s-1}(f)$$

for $N = M_s^{(a)}, M_s^{(a)} + 1, \ldots, N_s$. As before, $a = l$ if $j = 2i - 1$ and $a = r$ if $j = 2i$. Of course, for $N > N_{s-1}$, the subtracted sum above is absent.

Thus, on the interval $I_{i,s-1}$, we consider polynomials $e_{N,i,s-1}$ up to the degree N_{s-1}. The functional $\xi_{N_{s-1},i,s-1}$ is defined by $N_{s-1} + 1$ points, $N_{s-1}/2 + 1$ of them belong to the left subinterval $I_{2i-1,s}$. They are just the zeros of the first polynomial on this subinterval. The other $N_{s-1}/2$ points give the zeros of $e_{M_s^{(r)},2i,s}$. By the arguments in Section 2, we see that the system $(e, \eta) := (e_{N,j,s}, \eta_{N,j,s})_{s=0,\ j=1,\ N=M_s}^{\infty,\ 2^s,\ N_s}$ is biorthogonal with the total on the $\mathcal{E}(K(\Lambda))$ sequence of functionals. This satisfies the condition of the Dynin-Mityagin criterion, provided a suitable choice of the sequence $(n_s)_0^\infty$ is made.

Theorem 2. *Let $K(\Lambda)$ be a Cantor-type set. If a nondecreasing unbounded sequence $(N_s)_{s=0}^\infty$ of natural numbers of the form $N_s = 2^{n_s}$ is such that for some Q the sequence $(2^{N_s} l_s^Q)_{s=0}^\infty$ is bounded, then the system (e, η) is a basis in the space $\mathcal{E}(K(\Lambda))$.*

Proof. We can assume, by increasing Q if necessary, that for $s \geq 1$,

$$2^{N_s} l_s^Q \leq 1. \tag{4}$$

Let us take $p = 2^u$ and q of the form 2^v such that $q \geq p + 5Q + 1$. Fix s with $2^{n_{s-1}} > 4q$ and $j \leq 2^s$. Fix $\frac{1}{2}N_{s-1} \leq N \leq N_s$. Let $N = 2^n + \nu$ with $n_{s-1} - 1 \leq n \leq n_s$ and $0 \leq \nu < 2^n$. Then the function $e_{N,j,s}$ has zeros at all endpoints of the type less than or equal to $s + n - 1$ on $I_{j,s}$ (this is the set X_{r_0}, $r_0 = s + n$) and some endpoints of the type $s + n$ from other sets X_{r_k}, $s \leq r_k < r_0$. Analysis similar to that in the proof of Theorem 1 shows that

$$||e_{N,j,s}||_p \leq 5(3N)^p \prod_{p+1}^N z_k.$$

Here the nondecreasing set $(z_k)_1^N$ consists of the lengths $l_{s+n}, l_{s+n-1}, \dots, l_s$ taken from the product $l_{s+n} l_{s+n-1} l_{s+n-2}^2 \cdots l_s^{2^{n-1}}$, corresponding to the set X_{s+n}, with the similar products corresponding to the sets X_{r_k}. Note that the points from $K(\Lambda)\backslash I_{j,s}$ have no influence on the estimation of $\|e_{N,j,s}\|_p$ for $p < N$, since $\operatorname{dist}(I_{j,s}, K(\Lambda)\backslash I_{j,s}) = h_{s-1}$ is larger than l_s.

The task is now to examine the functional $\eta_{N,j,s}$. Without loss of generality, we can assume that j is even. Let $j = 2i$. The interval $I_{2i,s}$ is a subinterval of $I_{i,s-1}$. Therefore,

$$\eta_{N,2i,s} = \xi_{N,2i,s} - \sum_{k=N}^{N_{s-1}} \xi_{N,2i,s}(e_{k,i,s-1})\xi_{k,i,s-1}. \tag{5}$$

Repeating (2) and (3), we have

$$|\xi_{N,2i,s}(f)| \le C_q 2^{N_s} ||f||_r \prod\nolimits_N^{-1}, \tag{6}$$

where $\prod_N$ denotes the minimal product corresponding to the functional $\xi_{N,2i,s}$. This product contains $N-q$ terms of the type $|x_{n,2i,s} - x_{m,2i,s}|$.

As in the proof of Theorem 1, since $l_s \le 3h_s$, we get

$$\prod\nolimits_N^{-1} \le (l_s/h_s)^N \left(\prod_{q+1}^N z_k\right)^{-1} \le 3^{N_s} \left(\prod_{q+1}^N z_k\right)^{-1}. \tag{7}$$

Our claim is that the norm $|\cdot|_{-r}$ of the subtracted sum in (5) (and, consequently, of $\eta_{N,2i,s}$) can be estimated from above by the expression similar to the right-hand side of (7). Now $\frac{1}{2}N_{s-1} \le N \le N_{s-1}$.

First note that, for any k, $N \le k \le N_{s-1}$, we have

$$|\xi_{N,2i,s}(e_{k,i,s-1})| = \frac{|e_{k,i,s-1}^{(N)}(\theta)|}{N!} \le \binom{k}{N} l_{s-1}^{k-N} \le 2^k l_{s-1}^{k-N}. \tag{8}$$

If $N = N_{s-1}$, then $\eta_{N,2i,s} = \xi_{N,2i,s} - \xi_{N,i,s-1}$. Obviously, $|\xi_{N,i,s-1}|_{-r}$ has the desired bound. Hence, we can assume that $N+1 = \frac{1}{2}N_{s-1} + \nu$ with $1 \le \nu \le \frac{1}{2}N_{s-1}$. From $N+1$ points on $I_{2i,s}$ that define the functional $\xi_{N,2i,s}$ we have $2^{n_{s-1}-1}$ endpoints of the type less than or equal to $s + n_{s-1} - 2$ and ν (at least one is included!) endpoints of the type $s + n_{s-1} - 1$.

Fix k such that $N \le k \le N_{s-1}$. Denote by Z_k the set $(x_{n,i,s-1})_{n=1}^{k+1}$, which defines the functional $\xi_{k,i,s-1}$.

As in (6) we have the bound

$$|\xi_{k,i,s-1}|_{-r} \le C_q 2^k \prod\nolimits_k^{-1}, \tag{9}$$

where $\prod_k$ denotes the minimal product $\prod_{t=1}^{k-q} y_t$ corresponding to the functional $\xi_{k,i,s-1}$. The terms of $\prod_k$ are arranged in increasing order. The interval $I_{i,s-1}$ contains $2^{n_{s-1}}$ endpoints of the type $\le s + n_{s-1} - 2$. Therefore the chosen $k+1$ points occupy all endpoints of the type $\leq\ s + n_{s-1} - 3$ and some endpoints (maybe all for $k \geq N_{s-1} - 1$) of the type $\leq s + n_{s-1} - 2$. If $k = N_{s-1}$, then we get one endpoint of the type $s + n_{s-1} - 1$.

Suppose that k is even. Let $k = 2m$. Then the interval $I_{2i-1,s}$ contains $m+1$ points of Z_k, whereas $I_{2i,s}$ contains only m. Since $m \geq q$, we have to choose $q+1$ consecutive points from Z_k on $Z_k \cap I_{2i-1,s}$ in order to get the minimal value $\prod_k$. Consider the decomposition

$$\prod\nolimits_k = \prod_{t=1}^{m+1-q} y_t \cdot \prod_{t=m+2-q}^{N-q} y_t \cdot \prod_{t=N+1-q}^{k-q} y_t = \pi_1 \cdot \pi_2 \cdot \pi_3.$$

Since $m \leq N$, the value of π_1 is not smaller than the product of the first $m+1-q$ terms of $\prod_N$. In fact, π_1 is equal to this product only in the case when $N = \frac{1}{2} N_{s-1}$ and $k = N_{s-1}$. Then the configuration of points of $Z_k \cap I_{2i-1,s}$ completely repeats that of $Z_N := (x_{n,\,2i,\,s})_{n=1}^{N+1}$. In all other cases we have $m < N$ and the density of distribution of points from $Z_k \cap I_{2i-1,s}$ is smaller than the one for Z_N.

On the other hand, any term of π_2 is not smaller than $l_s + h_{s-1}$, so it is larger than any term of $\prod_N$. Hence, $\pi_1 \cdot \pi_2 > \prod_N$. Any term of π_3 is larger than h_{s-1}. Therefore,

$$\prod\nolimits_k \geq \prod\nolimits_N \cdot h_{s-1}^{k-N}.$$

The same conclusion can be drawn for $k = 2m+1$. Since N_{s-1} is even, we get $k \leq N_{s-1} - 1$. Then $m \leq \frac{1}{2} N_{s-1} - 1$ and so $m + 1 \leq N$.

Taking into account (8) and (9), we see that

$$|\xi_{N,\,2i,\,s}(e_{k,\,i,\,s-1})| \cdot |\xi_{k,\,i,\,s-1}|_{-r} \leq C_q\, 2^{2k} (l_{s-1}/h_{s-1})^{k-N} \cdot \prod\nolimits_N^{-1},$$

with $k - N \leq \frac{1}{2} N_{s-1}$. Substituting this and (6) in (5), we get

$$|\eta_{N,\,2i,\,s}|_{-r} \leq C_q \cdot [2^{N_s} + 1/2 \cdot N_{s-1} 2^{2\,N_{s-1}}\, 3^{(1/2)\,N_{s-1}}] \cdot \prod\nolimits_N^{-1}.$$

The expression in brackets is smaller than 10^{N_s}, as is easy to check. Applying (7) and (4) gives

$$|\eta_{N,\,2i,\,s}|_{-r} \leq C_q \cdot 30^{N_s} \left(\prod_{q+1}^{N} z_k \right)^{-1} \leq C_q l_s^{-5Q} \left(\prod_{q+1}^{N} z_k \right)^{-1}.$$

Therefore,

$$||e_{N,\,j,\,s}||_p \cdot |\eta_{N,2i,s}|_{-r} \leq 5 C_q 3^p N_s^p l_s^{-5Q} z_{p+1} \times \cdots \times z_q.$$

Replacing all z_k by l_s we get the bounded sequence on the right-hand side, due to the choice of q, and the proof is complete. ■

By means of Theorem 2 we get a variety of different bases in the space $\mathcal{E}(K(\Lambda))$. Any sequence $n_s \nearrow \infty$ with the bound $\overline{\lim}_s 2^{n_s}/\log_2 l_s^{-1} < \infty$, gives the basis property of the system (e, η).

Question. Are these bases quasi-equivalent, that is, equivalent after renumerating and multiplication by nonnull scalars?

It is a simple matter now to show bases in the spaces of Whitney functions on concrete Cantor-type sets. In the case of the classical Cantor ternary set, one can take, for example, $n_s = [\log_2 s]$ for $s \geq 2$. Here $[b]$ denotes the greatest integer in b. If $\alpha_s = \alpha,\ s \in \mathbf{N}$, for some $\alpha > 1$, then we get the compact set $K^{(\alpha)}$ from [7] or $K_2^{(\alpha)}$ from [1]. This has the extension property if and only if $\alpha \leq 2$. For any $\alpha > 1$ we can take $n_s = [(s-1)\ \log_2 \alpha]$ for $s \geq 1 + \ln 2/\ln \alpha$ in order to get (4). But if $\alpha \geq 2$ we can use as a basis the sequence $(e_N)_{N=0}^{\infty}$ from Theorem 1 as well.

The restriction $3l_{s+1} \leq l_s$ at the beginning of Section 3 is essential for the estimation of the dual norms of the functionals ξ_N and $\xi_{N,j,s}$. One may conjecture that the method suggested can also be applied in the case $\exists\, \varepsilon_0 :\ (2 + \varepsilon_0) l_{s+1} \leq l_s, s \in \mathbf{N}$, but with another sequence (x_n), more closely related to the structure of the set $K(\Lambda)$. On the other hand, the condition $\exists\, C :\ C\, l_{s+1} \geq l_s,\ s \in \mathbf{N}$, gives the uniformly perfect compact set $K(\Lambda)$ with the Markov property.

A slight change in the proof gives the basis in the spaces of Whitney functions on the sets $K_N^{(\alpha)}$ and, moreover, in the more general case $K((l_s), (N_s))$ with $N_s \leq N,\ s \in \mathbf{N}$ (see [1] for the definition). However, the question in [1] about the existence of a basis in the space $\mathcal{E}(K_\infty)$ remains open if K_∞ does not have the extension property.

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